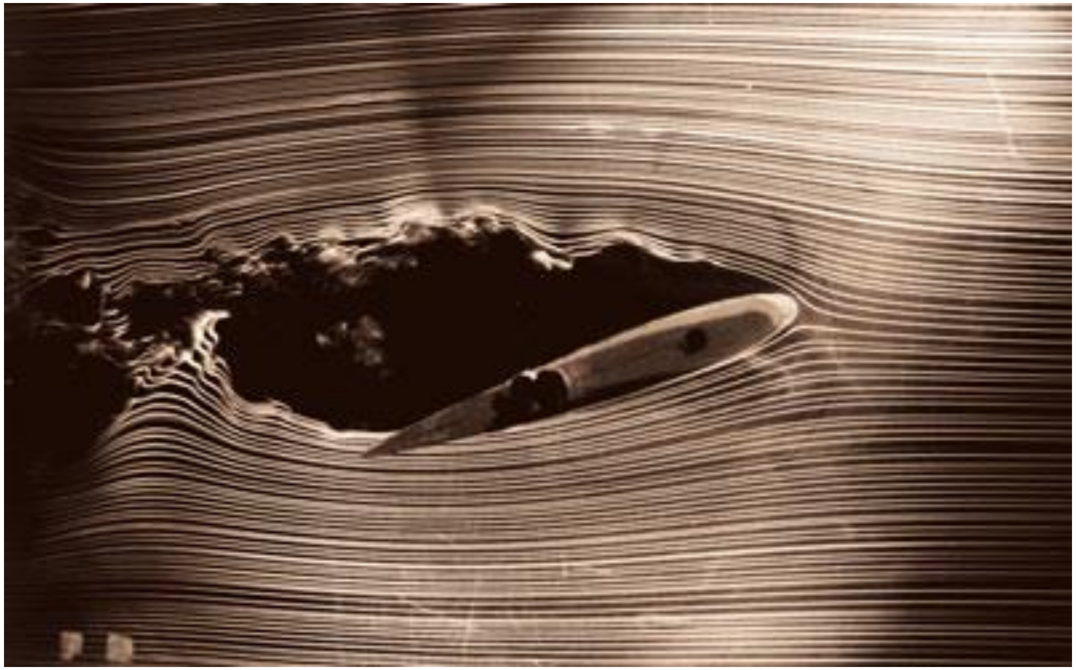
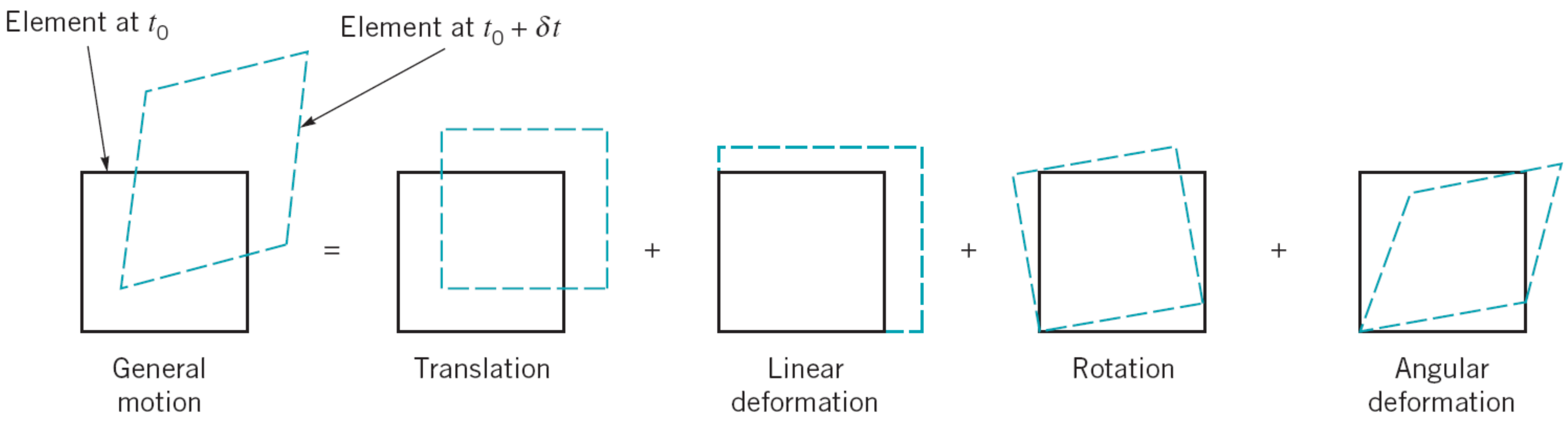
# Differential analysis of fluid flow



Sometimes the control volume of interest is infinitesimally small
(a point in space rather than to a 2D or 3D volume)

→ Differential analysis rather than finite control volume analysis

# Types of motion and deformation for a fluid element.



# Velocity and acceleration field

Velocity: $\vec{V} = u\vec{i} + v\vec{j} + w\vec{k}$

Acceleration: $\vec{a} = \frac{\partial \vec{V}}{\partial t} + u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}$

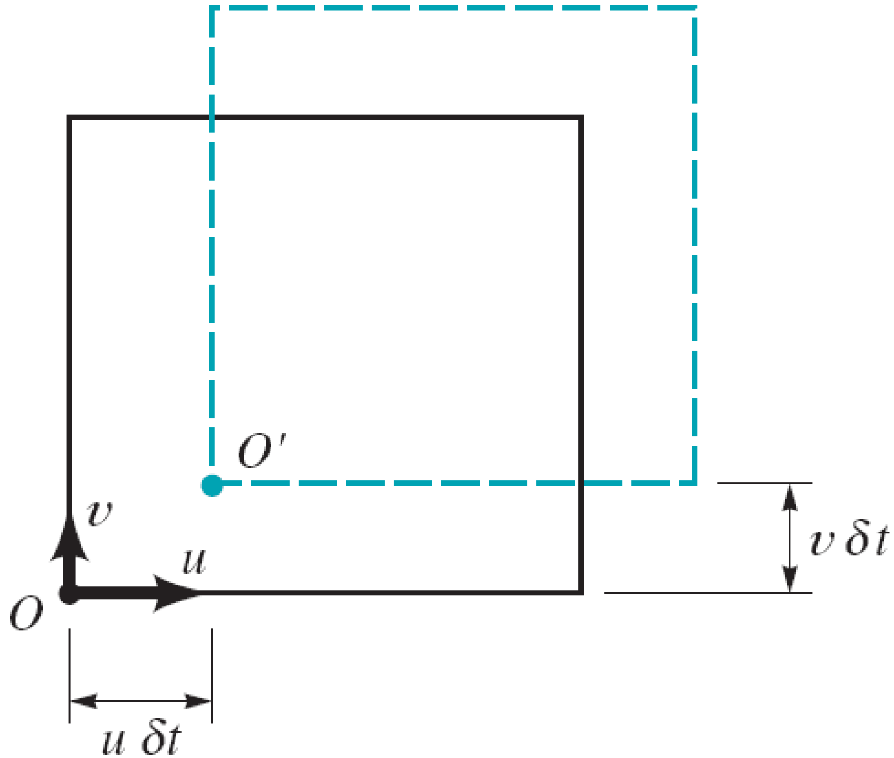
The acceleration is concisely expressed as: $\vec{a} = \frac{D\vec{V}}{Dt}$

where the operator $\frac{D()}{Dt} = \frac{\partial()}{\partial t} + u\frac{\partial()}{\partial x} + v\frac{\partial()}{\partial y} + w\frac{\partial()}{\partial z}$

is termed **material derivative.** In vector notation: $\frac{D()}{Dt} = \frac{\partial()}{\partial t} + (\vec{V} \bullet \vec{\nabla})()$

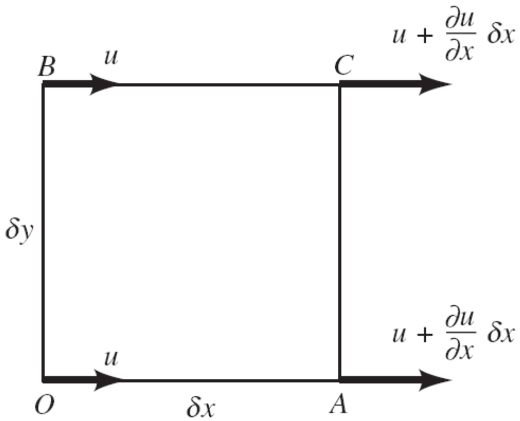
where the gradient operator is $\vec{\nabla}() = \frac{\partial()}{\partial x}\vec{i} + \frac{\partial()}{\partial y}\vec{j} + \frac{\partial()}{\partial z}\vec{k}$

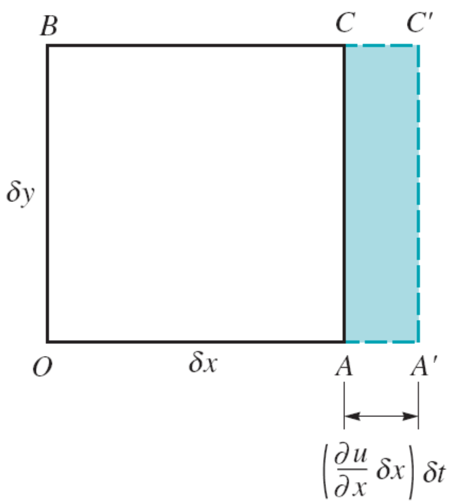
# Translation of a fluid element.




If the velocity, *V*, is the same for all fluid elements, we have **translation** without deformation

# Linear deformation of a fluid element.




If <u>velocity gradients</u> are present

$\rightarrow$ deformation

Fluid element volume: $\delta\forall = \delta x\,\delta y\,\delta z$

Change in $\delta\forall = \left[\left(\frac{\partial u}{\partial x}\delta x\right)\delta t\right]\delta y\,\delta z$

Rate of change of volume per unit volume:

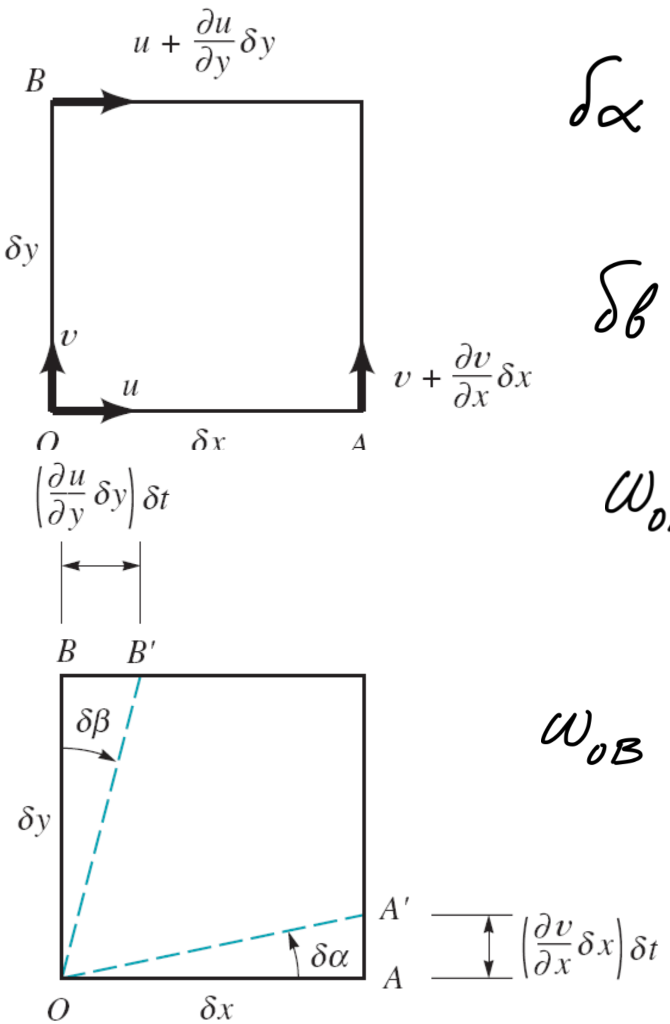
$$\frac{1}{\delta\forall}\frac{d(\delta\forall)}{dt} = \frac{1}{\delta x\,\delta y\,\delta z}\lim_{\delta t\to 0}\frac{\frac{\partial u}{\partial x}\delta x\cdot\delta y\cdot\delta z\;\delta t}{\delta t} = \frac{\partial u}{\partial x}$$

In general,

$$\frac{1}{\delta\forall}\frac{d(\delta\forall)}{dt} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \vec{\nabla}\cdot\vec{V}$$

Volumetric dilatation rate

# Angular motion and deformation of a fluid element.




$$\delta\alpha \simeq \tan\alpha = \frac{\frac{\partial v}{\partial x}\delta x\,\delta t}{\delta x} = \frac{\partial v}{\partial x}\delta t$$

$$\delta\beta \simeq \tan\beta = \frac{\frac{\partial u}{\partial y}\delta y\,\delta t}{\delta y} = \frac{\partial u}{\partial y}\delta t$$

$$\omega_{OA} = \lim_{\delta t \to 0} \frac{\delta\alpha}{\delta t} = \frac{\partial v}{\partial x}$$

if $\frac{\partial v}{\partial x} > 0 \rightsquigarrow$ CCW

$$\omega_{OB} = \lim_{\delta t \to 0} \frac{\delta\beta}{\delta t} = \frac{\partial u}{\partial y}$$

if $\frac{\partial u}{\partial y} > 0 \rightsquigarrow$ CW

By convention:
CCW is <u>positive</u>

# Rotation vector and vorticity

Definition of <u>rotation</u>: $\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$

Similarly: $\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)$

$\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$

$$\vec{\omega} = \omega_x \vec{i} + \omega_y \vec{j} + \omega_z \vec{k}$$

$$\vec{\omega} = \frac{1}{2} \operatorname{curl} \vec{V} = \frac{1}{2} \vec{\nabla} \times \vec{V}$$

$$\frac{1}{2} \vec{\nabla} \times \vec{V} = \frac{1}{2} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$

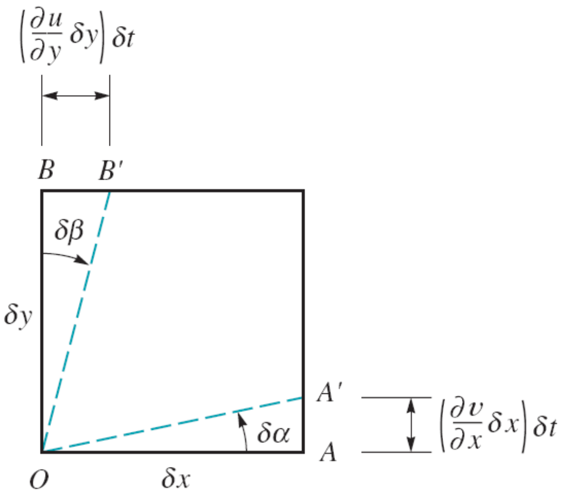
$$= \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\hat{i} + \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\hat{j} + \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{k}$$

Define <u>vorticity</u> $\vec{\zeta} = 2\vec{\omega} = \vec{\nabla} \times \vec{V}$

**Remarks:** a) if $\omega_{OA} = -\omega_{OB}$ or $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$\rightsquigarrow$ rotation as an <u>undeformed block</u>;

otherwise: <u>angular</u> <u>deformation</u>




b) When $\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$

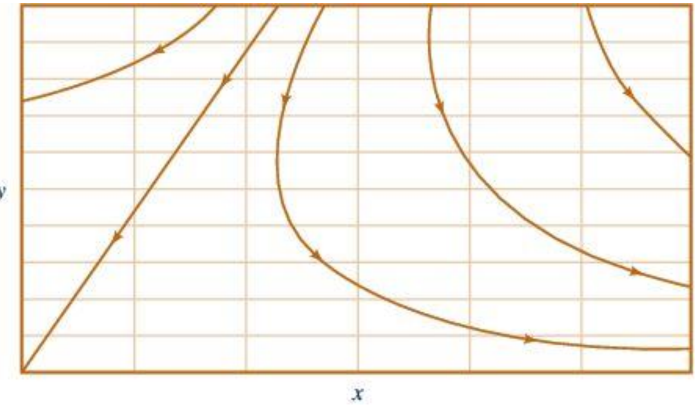
$$\omega_z = \left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) = 0$$

rotation around z-axis is zero

In general, when $\vec{\nabla} \times \vec{V} = 0 \rightarrow \vec{\omega} = \vec{\zeta} = 0$

$\rightarrow$ <u>Irrotational flow</u>

# Example: vorticity




Flow field with $u = 4xy$
$v = 2(x^2 - y^2)$
$w = 0$

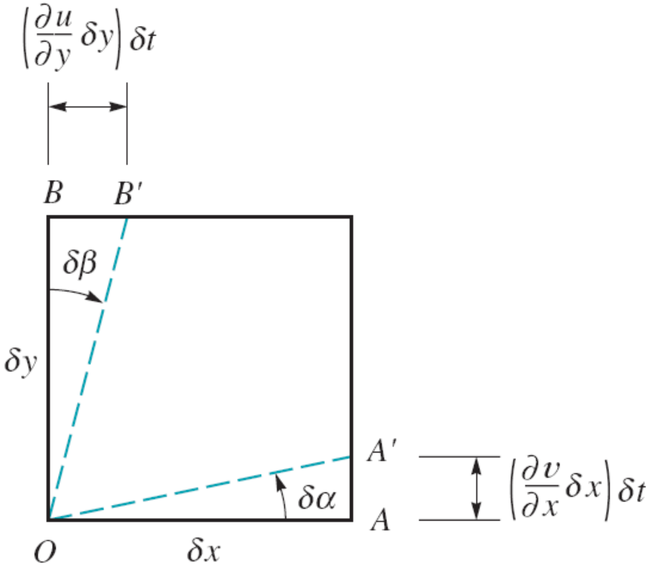
Is the flow field irrotational?

This is a 2-D flow field ($w=0$). Hence
$\omega_x = \omega_y = 0$ (Check!)

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \frac{1}{2}\left(4x - 4x\right) = 0$$

The flow field is irrotational.

# Angular deformation and shearing strain




<u>Rate of angular deformation</u>

The change in the original right angle (90°) between OA and OB, $\delta\gamma$

$$\delta\gamma = \delta\alpha + \delta\beta$$

Rate of change of $\delta\gamma$ or <u>shearing strain</u>, $\dot{\gamma}$, is

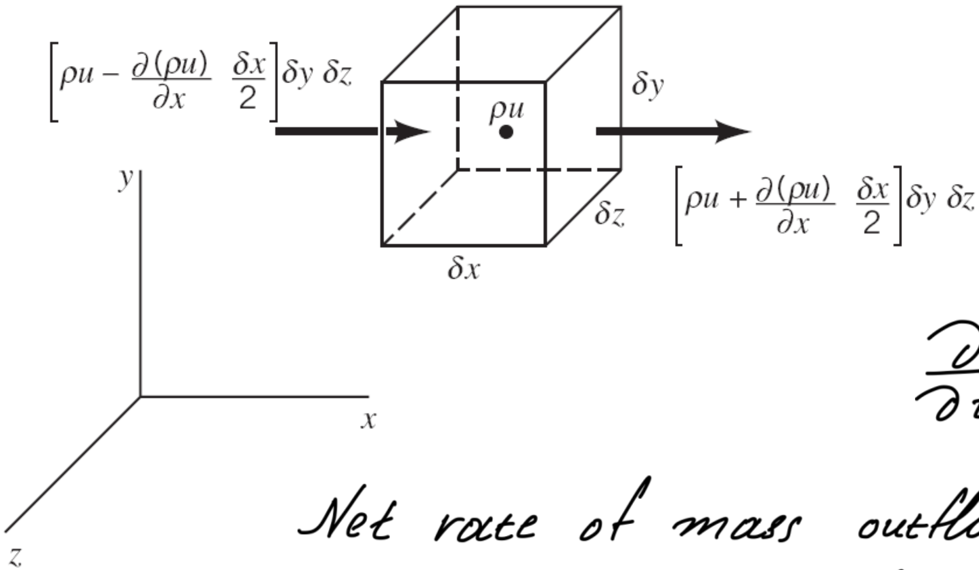
$$\lim_{\delta t \to 0} \frac{\delta\gamma}{\delta t} = \lim_{\delta t \to 0} \left[ \frac{\frac{\partial v}{\partial x}\delta t + \frac{\partial u}{\partial y}\delta t}{\delta t} \right]$$

$$\Rightarrow \quad \dot{\gamma} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

# Differential form of the conservation of mass equation.

Conservation of mass: $$\frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V}\cdot\vec{n}\, dA = 0 \qquad (1)$$

## Differential form of the continuity equation




$\rho u$ = mass flow per unit area

$$\frac{\partial}{\partial t}\int_{CV} \rho \, d\forall \simeq \frac{\partial \rho}{\partial t}\,\delta x\, \delta y\, \delta z \qquad (2)$$

Net rate of mass outflow in x-direction =

= rate of mass outflow − rate of mass inflow

$$= \left[\rho u + \frac{\partial(\rho u)}{\partial x}\frac{\delta x}{2}\right]\delta y\, \delta z - \left[\rho u - \frac{\partial(\rho u)}{\partial x}\frac{\delta x}{2}\right]\delta y\, \delta z$$

$$= \frac{\partial(\rho u)}{\partial x}\,\delta x\, \delta y\, \delta z$$

x-direction: $\dfrac{\partial(\rho u)}{\partial x}\,\delta x\,\delta y\,\delta z$

y-direction: $\dfrac{\partial(\rho v)}{\partial y}\,\delta x\,\delta y\,\delta z$

z-direction: $\dfrac{\partial(\rho w)}{\partial z}\,\delta x\,\delta y\,\delta z$

Net rate of mass outflow:

$$\left[\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}\right]\delta x\,\delta y\,\delta z \qquad (3)$$

(1), (2) & (3) $\Rightarrow$

$$\frac{\partial \rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}=0$$

Conservation of mass or continuity

In vector notation:

$$\frac{\partial \rho}{\partial t}+\vec{\nabla}\cdot(\rho\vec{V})=0$$

Special cases: 1) Steady flow $\vec{\nabla}\cdot(\rho\vec{V})=0$

or $\dfrac{\partial(\rho u)}{\partial x}+\dfrac{\partial(\rho v)}{\partial y}+\dfrac{\partial(\rho w)}{\partial z}=0$

2) Incompressible flow ($\rho = ct$)

$\vec{\nabla}\cdot\vec{V}=0$ or $\dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial y}+\dfrac{\partial w}{\partial z}=0$

# Example: 3D steady, incompressible flow

Velocity components in a steady and incompressible flow field:

$$u = x^2 + y^2 + z^2$$

$$v = xy + yz + z$$

$$w = ?$$

Solution: To satisfy the continuity equation for steady and incompressible flow:

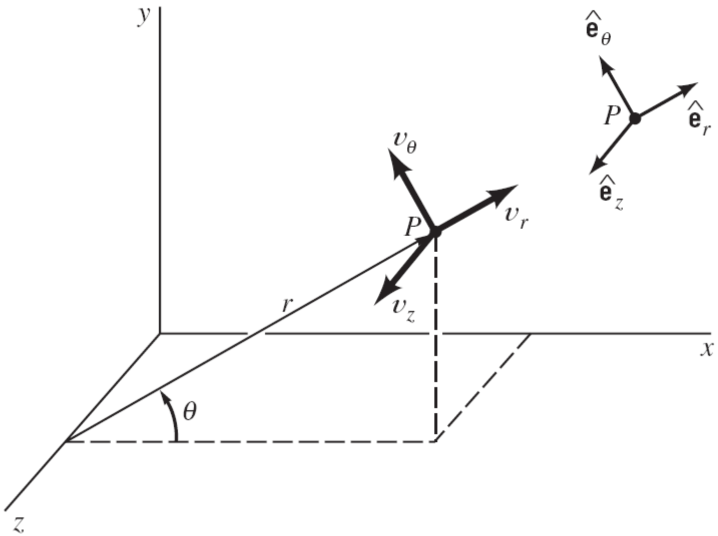
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\Rightarrow \quad 2x + (x+z) + \frac{\partial w}{\partial z} = 0$$

$$\Rightarrow \quad \frac{\partial w}{\partial z} = -2x - (x+z) = -3x - z$$

$$\Rightarrow \quad w = -3xz - \frac{1}{2}z^2 + \underbrace{f(x,y)}_{\text{need more info to define}}$$

# Velocity components in cylindrical polar coordinates.




Continuity eq. in cylindrical coordinates:

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial (r \rho v_r)}{\partial r} + \frac{1}{r}\frac{\partial (\rho v_\theta)}{\partial \theta} + \frac{\partial (\rho v_z)}{\partial z} = 0$$

For incompressible flow (steady or unsteady):

$$\frac{1}{r}\frac{\partial (r v_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0$$

# Stream function

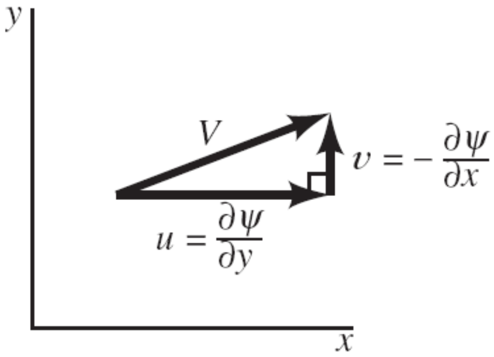
For 2-D flow $\vec{V} = u\hat{i} + v\hat{j}$

For steady, incompressible, 2-D flow, the continuity equation is:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Define stream function $\psi(x,y)$ such as

$$u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x}$$




Continuity:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial x}\right)$$

$$= \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial x \partial y} \equiv 0$$

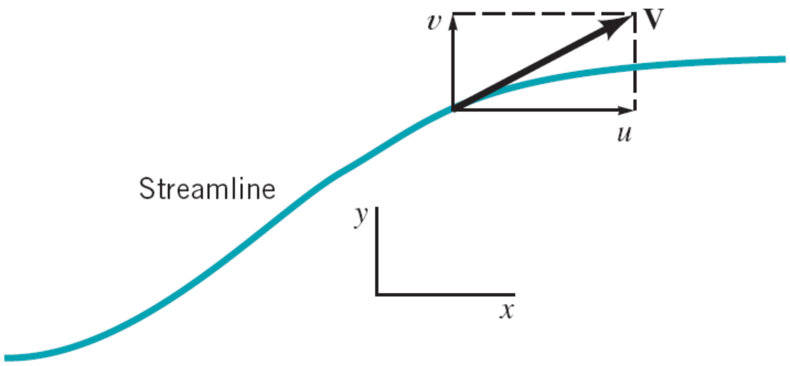
$\therefore$ When $\psi$ exists $\Rightarrow$ continuity is satisfied

# Stream function property #1

Lines along which $\psi = ct$ are <u>streamlines</u>

Streamline: line tangent to $\vec{V}$

$$\frac{dy}{dx} = \frac{v}{u}$$




$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy$$

$$= -v\,dx + u\,dy$$

When $\psi = ct \Rightarrow d\psi = 0$

$$\Rightarrow -v\,dx + u\,dy = 0$$

$$\Rightarrow \underline{\underline{\frac{dy}{dx} = \frac{v}{u}}} \quad \text{streamline!}$$

Plot $\psi(x,y)$ to <u>visualize</u> the flow field

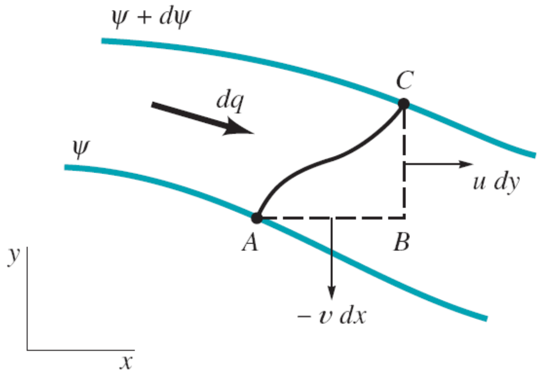
Note: $\psi$ defined within a constant.

# Stream function property #2

Change in the value of $\psi$ between two streamlines equals the volume rate of flow

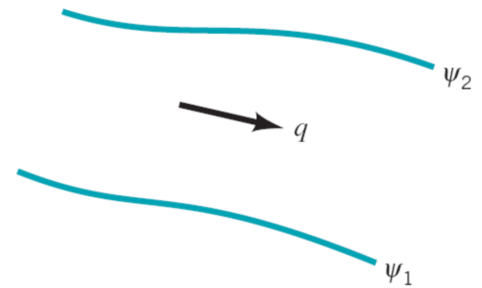
Flow ~~never~~ crosses streamlines $\vec{V} \parallel \psi$

$$\underbrace{dq}_{\text{inflow}} = \underbrace{-v\,dx + u\,dy}_{\text{outflow}}$$




$$\Rightarrow \quad dq = \frac{\partial \psi}{\partial x}\,dx + \frac{\partial \psi}{\partial y}\,dy = d\psi$$

$$\Rightarrow \quad q = \int_{\psi_1}^{\psi_2} d\psi = \psi_2 - \psi_1$$




Note: if $\psi_2 > \psi_1 \rightarrow$ flow from left to right

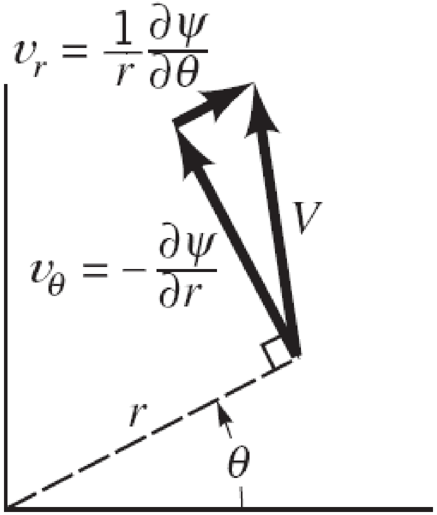
if $\psi_2 < \psi_1 \rightarrow$ flow from right to left

# Stream function in cylindrical coordinates

In cylindrical coordinates:

Continuity: $\dfrac{1}{r}\dfrac{\partial(r v_r)}{\partial r} + \dfrac{1}{r}\dfrac{\partial v_\theta}{\partial \theta} = 0$

$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \qquad v_\theta = -\frac{\partial \psi}{\partial r}$$



# Example: stream function

Steady, incompressible 2-D flow with:

$u = 2y$

$v = 4x$

a) $\psi = ?$

b) Streamlines?

c) Direction of flow

$$u = \frac{\partial \psi}{\partial y} = 2y \Rightarrow \psi = y^2 + f(x) \qquad (1)$$

$$v = -\frac{\partial \psi}{\partial x} = 4x \Rightarrow \frac{\partial \psi}{\partial x} = -4x \qquad (2)$$

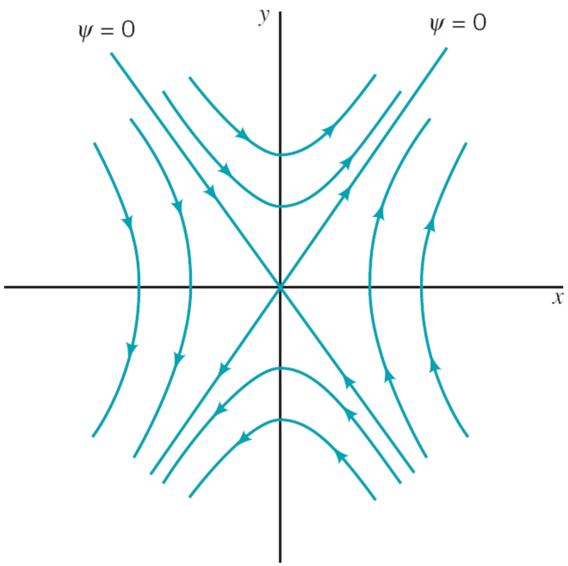
$$(1) \wedge (2) \Rightarrow \frac{df}{dx} = -4x \Rightarrow f = -2x^2 + c \qquad (3)$$

$c$ is arbitrary; let $c = 0$ for simplicity

$$(1) \wedge (3) \Rightarrow \underline{\underline{\psi = y^2 - 2x^2}}$$

For $\psi = 0 \Rightarrow y^2 - 2x^2 = 0 \Rightarrow y = \pm\sqrt{2}\, x$

In general, $\underline{\underline{\dfrac{y^2}{\psi} - \dfrac{x^2}{\psi/2} = 1}}$ hyperbola




Direction of flow:

$v = 4x$

So, for $\underline{\underline{x > 0}}$ flow is $\underline{\underline{\text{upwards}}}$